MODULE 3

Extended Surface Heat Transfer

3.1 Introduction:

Convection: Heat transfer between a solid surface and a moving fluid is governed by the Newton's cooling law: $q = hA(T_s - T_\infty)$, where T_s is the surface temperature and T_∞ is the fluid temperature. Therefore, to increase the convective heat transfer, one can

- Increase the temperature difference $(T_s - T_\infty)$ between the surface and the fluid.
- Increase the convection coefficient h. This can be accomplished by increasing the fluid flow over the surface since h is a function of the flow velocity and the higher the velocity, the higher the h. Example: a cooling fan.
- Increase the contact surface area A. Example: a heat sink with fins.

Many times, when the first option is not in our control and the second option (i.e. increasing h) is already stretched to its limit, we are left with the only alternative of increasing the effective surface area by using fins or extended surfaces. Fins are protrusions from the base surface into the cooling fluid, so that the extra surface of the protrusions is also in contact with the fluid. Most of you have encountered cooling fins on air-cooled engines (motorcycles, portable generators, etc.), electronic equipment (CPUs), automobile radiators, air conditioning equipment (condensers) and elsewhere.

3.2 Extended surface analysis:

In this module, consideration will be limited to steady state analysis of rectangular or pin fins of constant cross sectional area. Annular fins or fins involving a tapered cross section may be analyzed by similar methods, but will involve solution of more complicated equations which result. Numerical methods of integration or computer programs can be used to advantage in such cases.

We start with the General Conduction Equation:

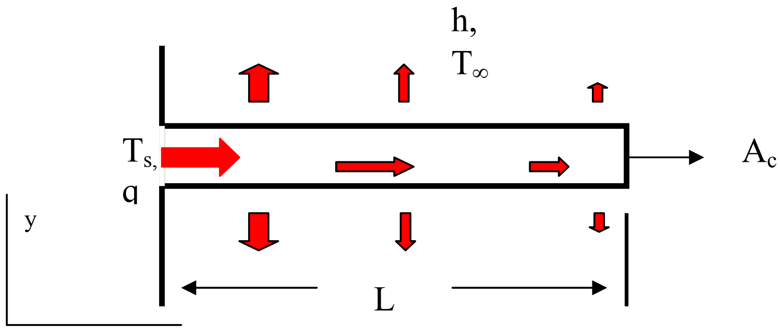
$$\frac{1}{\alpha} \cdot \left.\frac{dT}{d\tau}\right|_{system} = \nabla^2 T + \frac{\ddot{q}}{k} \qquad (1)$$

After making the assumptions of Steady State, One-Dimensional Conduction, this equation reduces to the form:

$$\frac{d^2T}{dx^2} + \frac{\ddot{q}}{k} = 0 \qquad (2)$$

This is a second order, ordinary differential equation and will require 2 boundary conditions to evaluate the two constants of integration that will arise.

Consider the cooling fin shown below:

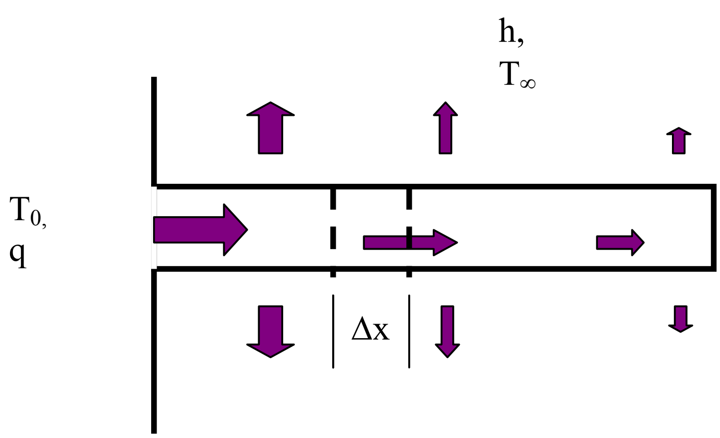


The fin is situated on the surface of a hot surface at T_s and surrounded by a coolant at temperature T_∞, which cools with convective coefficient, h. The fin has a cross sectional area, A_c, (This is the area through with heat is conducted.) and an overall length, L.

Note that as energy is conducted down the length of the fin, some portion is lost, by convection, from the sides. Thus the heat flow varies along the length of the fin.

We further note that the arrows indicating the direction of heat flow point in both the x and y directions. This is an indication that this is truly a two- or three-dimensional heat flow, depending on the geometry of the fin. However, quite often, it is convenient to analyse a fin by examining an equivalent one–dimensional system. The equivalent system will involve the introduction of heat sinks (negative heat sources), which remove an amount of energy equivalent to what would be lost through the sides by convection.

Consider a differential length of the fin.



Across this segment the heat loss will be $h\cdot(P\cdot\Delta x)\cdot(T-T_\infty)$, where P is the perimeter around the fin. The equivalent heat sink would be $\ddot{q}\cdot\left(A_c\cdot\Delta x\right)$.

Equating the heat source to the convective loss:

$$\ddot{q} = \frac{-h \cdot P \cdot \left(T - T_\infty\right)}{A_c} \tag{3}$$

Substitute this value into the General Conduction Equation as simplified for One-Dimension, Steady State Conduction with Sources:

$$\frac{d^2T}{dx^2} - \frac{h \cdot P}{k \cdot A_c} \cdot \left(T - T_\infty\right) = 0 \tag{4}$$

which is the equation for a fin with a constant cross sectional area. This is the Second Order Differential Equation that we will solve for each fin analysis. Prior to solving, a couple of simplifications should be noted. First, we see that h, P, k and A_c are all independent of x in the defined system (They may not be constant if a more general analysis is desired.). We replace this ratio with a constant. Let

$$m^2 = \frac{h \cdot P}{k \cdot A_c} \tag{5}$$

then:

$$\frac{d^2T}{dx^2} - m^2 \cdot \left(T - T_\infty\right) = 0 \tag{6}$$

Next we notice that the equation is non-homogeneous (due to the T_∞ term). Recall that non-homogeneous differential equations require both a general and a particular solution. We can make this equation homogeneous by introducing the temperature relative to the surroundings:

$$\theta \equiv T - T_\infty \tag{7}$$

Differentiating this equation we find:

$$\frac{d\theta}{dx} = \frac{dT}{dx} + 0 \tag{8}$$

Differentiate a second time:

$$\frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2} \tag{9}$$

Substitute into the Fin Equation:

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \tag{10}$$

This equation is a Second Order, Homogeneous Differential Equation.

3.3 Solution of the Fin Equation

We apply a standard technique for solving a second order homogeneous linear differential equation.

Try $\theta = e^{\alpha \cdot x}$. Differentiate this expression twice:

$$\frac{d\theta}{dx} = \alpha \cdot e^{\alpha \cdot x} \tag{11}$$

$$\frac{d^2\theta}{dx^2} = \alpha^2 \cdot e^{\alpha \cdot x} \tag{12}$$

Substitute this trial solution into the differential equation:

$$\alpha^2 \cdot e^{\alpha \cdot x} - m^2 \cdot e^{\alpha \cdot x} = 0 \tag{13}$$

Equation (13) provides the following relation:

$$\alpha = \pm\ m \tag{14}$$

We now have two solutions to the equation. The general solution to the above differential equation will be a linear combination of each of the independent solutions.

Then:

$$\theta = A \cdot e^{m \cdot x} + B \cdot e^{-m \cdot x} \tag{15}$$

where A and B are arbitrary constants which need to be determined from the boundary conditions. Note that it is a 2nd order differential equation, and hence we need two boundary conditions to determine the two constants of integration.

An alternative solution can be obtained as follows: Note that the hyperbolic sin, sinh, the hyperbolic cosine, cosh, are defined as:

$$\sinh(m \cdot x) = \frac{e^{m \cdot x} - e^{-m \cdot x}}{2} \qquad \cosh(m \cdot x) = \frac{e^{m \cdot x} + e^{-m \cdot x}}{2} \tag{16}$$

We may write:

$$C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) = C \cdot \frac{e^{m \cdot x} + e^{-m \div x}}{2} + D \cdot \frac{e^{m \cdot x} - e^{-m \div x}}{2} = \frac{C + D}{2} \cdot e^{m \cdot x} + \frac{C - D}{2} \cdot e^{-m \cdot x} \tag{17}$$

We see that if (C+D)/2 replaces A and (C-D)/2 replaces B then the two solutions are equivalent.

$$\theta = C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) \tag{18}$$

Generally the exponential solution is used for very long fins, the hyperbolic solutions for other cases.

Boundary Conditions:

Since the solution results in 2 constants of integration we require 2 boundary conditions. The first one is obvious, as one end of the fin will be attached to a hot surface and will come into thermal equilibrium with that surface. Hence, at the fin base,

$$\theta(0) = T_0 - T_\infty \equiv \theta_0 \tag{19}$$

The second boundary condition depends on the condition imposed at the other end of the fin. There are various possibilities, as described below.

Very long fins:
For very long fins, the end located a long distance from the heat source will approach the temperature of the surroundings. Hence,

$$\theta(\infty) = 0 \tag{20}$$

Substitute the second condition into the exponential solution of the fin equation:

$$\theta(\infty) = 0 = A\cdot \overset{\infty}{e^{m\cdot\infty}} + B\cdot \overset{0}{e^{-m\cdot\infty}} \tag{21}$$

The first exponential term is infinite and the second is equal to zero. The only way that this equation can be valid is if A = 0. Now apply the second boundary condition.

$$\theta(0) = \theta_0 = B\cdot e^{-m\cdot 0} \Rightarrow B = \theta_0 \tag{22}$$

The general temperature profile for a very long fin is then:

$$\theta(x) = \theta_0 \cdot e^{-m\cdot x} \tag{23}$$

If we wish to find the heat flow through the fin, we may apply Fourier Law:

$$q = -k\cdot A_c \cdot \frac{dT}{dx} = -k\cdot A_c \cdot \frac{d\theta}{dx} \tag{24}$$

Differentiate the temperature profile:

$$\frac{d\theta}{dx} = -\theta_o \cdot m\cdot e^{-m\cdot x} \tag{25}$$

So that:

$$q = k\cdot A_c \cdot \theta_0 \cdot \left[\frac{h\cdot P}{k\cdot A_c}\right]^{1/2} \cdot e^{-m\cdot x} = \sqrt{h\cdot P\cdot k\cdot A_c}\cdot e^{-m\cdot x}\cdot \theta_0 = M\theta_0 e^{-mx} \tag{26}$$

where $M = \sqrt{hPkA_c}$.

Often we wish to know the total heat flow through the fin, i.e. the heat flow entering at the base (x=0).

$$q = \sqrt{h \cdot P \cdot k \cdot A_c} \cdot \theta_0 = M\theta_0 \tag{27}$$

The insulated tip fin
Assume that the tip is insulated and hence there is no heat transfer:

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0 \tag{28}$$

The solution to the fin equation is known to be:

$$\theta = C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) \tag{29}$$

Differentiate this expression.

$$\frac{d\theta}{dx} = C \cdot m \cdot \sinh(m \cdot x) + D \cdot m \cdot \cosh(m \cdot x) \tag{30}$$

Apply the first boundary condition at the base:

$$\theta(0) = \theta_0 = C \underbrace{\sinh(m \cdot 0)}_{0} + D \underbrace{\cosh(m \cdot 0)}_{1} \tag{31}$$

So that D = θ_0. Now apply the second boundary condition at the tip to find the value of C:

$$\frac{d\theta}{dx}(L) = 0 = Cm \sinh(m \cdot L) + \theta_0 m \cosh(m \cdot L) \tag{32}$$

which requires that

$$C = -\theta_0 \frac{\cosh(mL)}{\sinh(mL)} \tag{33}$$

This leads to the general temperature profile:

$$\theta(x) = \theta_0 \frac{\cosh m(L - x)}{\cosh(mL)} \tag{34}$$

We may find the heat flow at any value of x by differentiating the temperature profile and substituting it into the Fourier Law:

$$q = -k \cdot A_c \cdot \frac{dT}{dx} = -k \cdot A_c \cdot \frac{d\theta}{dx} \tag{35}$$

So that the energy flowing through the base of the fin is:

$$q = \sqrt{hPkA_c}\,\theta_0 \tanh(mL) = M\theta_0 \tanh(mL) \qquad (36)$$

If we compare this result with that for the very long fin, we see that the primary difference in form is in the hyperbolic tangent term. That term, which always results in a number equal to or less than one, represents the reduced heat loss due to the shortening of the fin.

Other tip conditions:
We have already seen two tip conditions, one being the long fin and the other being the insulated tip. Two other possibilities are usually considered for fin analysis: (i) a tip subjected to convective heat transfer, and (ii) a tip with a prescribed temperature. The expressions for temperature distribution and fin heat transfer for all the four cases are summarized in the table below.

Table 3.1

Case	Tip Condition	Temp. Distribution	Fin heat transfer
A	Convection heat transfer: $h\theta(L)=-k(d\theta/dx)_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	$M\theta_o \dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$
B	Adiabatic $(d\theta/dx)_{x=L}=0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	$M\theta_0 \tanh mL$
C	Given temperature: $\theta(L)=\theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh m(L-x) + \sinh m(L-x)}{\sinh mL}$	$M\theta_0 \dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$
D	Infinitely long fin $\theta(L)=0$	e^{-mx}	$M\ \theta_0$

3.4 Fin Effectiveness

How effective a fin can enhance heat transfer is characterized by the fin effectiveness, ε_f, which is as the ratio of fin heat transfer and the heat transfer without the fin. For an adiabatic fin:

$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{\sqrt{hPkA_C}\tanh(mL)}{hA_C} = \sqrt{\frac{kP}{hA_C}}\tanh(mL) \qquad (37)$$

If the fin is long enough, mL>2, tanh(mL)→1, and hence it can be considered as infinite fin (case D in Table 3.1). Hence, for long fins,

$$\varepsilon_f \rightarrow \sqrt{\frac{kP}{hA_C}} = \sqrt{\left(\frac{k}{h}\right)\frac{P}{A_C}} \tag{38}$$

In order to enhance heat transfer, ε_f should be greater than 1 (In case ε_f <1, the fin would have no purpose as it would serve as an insulator instead). However $\varepsilon_f \geq 2$ is considered unjustifiable because of diminishing returns as fin length increases.

To increase ε_f, the fin's material should have higher thermal conductivity, k. It seems to be counterintuitive that the lower convection coefficient, h, the higher ε_f. Well, if h is very high, it is not necessary to enhance heat transfer by adding heat fins. Therefore, heat fins are more effective if h is low.

Observations:

- If fins are to be used on surfaces separating gas and liquid, fins are usually placed on the gas side. (Why?)
- P/A_C should be as high as possible. Use a square fin with a dimension of W by W as an example: P=4W, $A_C=W^2$, $P/A_C=(4/W)$. The smaller the W, the higher is the P/A_C, and the higher the ε_f.Conclusion: It is preferred to use thin and closely spaced (to increase the total number) fins.

  The effectiveness of a fin can also be characterized by

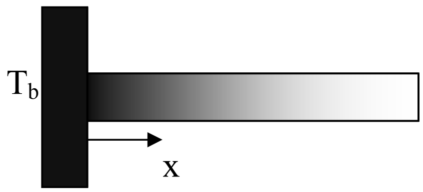
$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{(T_b - T_\infty)/R_{t,f}}{(T_b - T_\infty)/R_{t,h}} = \frac{R_{t,h}}{R_{t,f}} \tag{39}$$

  It is a ratio of the thermal resistance due to convection to the thermal resistance of a fin. In order to enhance heat transfer, the fin's resistance should be lower than the resistance due only to convection.

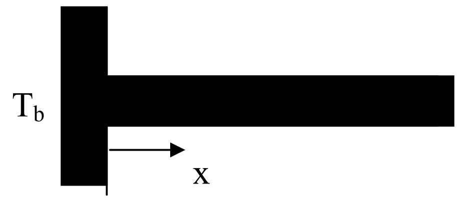
3.5 Fin Efficiency

The fin efficiency is defined as the ratio of the energy transferred through a real fin to that transferred through an ideal fin. An ideal fin is thought to be one made of a perfect or infinite conductor material. A perfect conductor has an infinite thermal conductivity so that the entire fin is at the base material temperature.

$$\eta = \frac{q_{real}}{q_{ideal}} = \frac{\sqrt{h \cdot P \cdot k \cdot A_c} \cdot \theta_L \cdot \tanh(m \cdot L)}{h \cdot (P \cdot L) \cdot \theta_L} \tag{40}$$



Real situation



Ideal situation

Simplifying equation (40):

$$\eta = \sqrt{\frac{k \cdot A_c}{h \cdot P}} \frac{\theta_L \cdot \tanh(m \cdot L)}{L \cdot \theta_L} = \frac{\tanh(m \cdot L)}{m \cdot L} \tag{41}$$

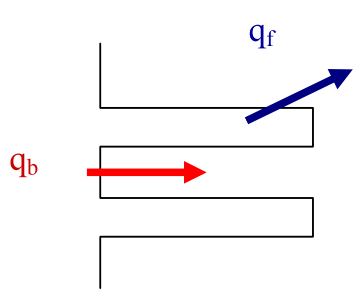
The heat transfer through any fin can now be written as:

$$q.\left[\frac{1}{\eta.h.A_f}\right] = (T - T_\infty \tag{42}$$

The above equation provides us with the concept of fin thermal resistance (using electrical analogy) as

$$R_{t,f} = \frac{1}{\eta.h.A_f} \tag{43}$$

Overall Fin Efficiency:
Overall fin efficiency for an array of fins



Define terms: A_b: base area exposed to coolant

A_f: surface area of a single fin

A_t: total area including base area and total finned surface, A_t=A_b+NA_f

N: total number of fins

Heat Transfer from a Fin Array:

$$q_t = q_b + Nq_f = hA_b(T_b - T_\infty) + N\eta_f hA_f(T_b - T_\infty)$$
$$= h[(A_t - NA_f) + N\eta_f A_f](T_b - T_\infty) = h[A_t - NA_f(1-\eta_f)](T_b - T_\infty)$$
$$= hA_t[1 - \frac{NA_f}{A_t}(1-\eta_f)](T_b - T_\infty) = \eta_O hA_t(T_b - T_\infty)$$

Define overall fin efficiency: $\eta_O = 1 - \frac{NA_f}{A_t}(1-\eta_f)$

$q_t = hA_t\eta_O(T_b - T_\infty) = \frac{T_b - T_\infty}{R_{t,O}}$ where $R_{t,O} = \frac{1}{hA_t\eta_O}$

Compare to heat transfer without fins

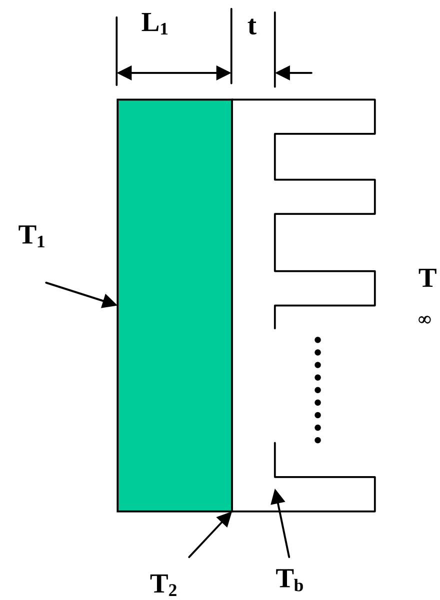
$q = hA(T_b - T_\infty) = h(A_b + NA_{b,f})(T_b - T_\infty) = \frac{1}{hA}$

where $A_{b,f}$ is the base area (unexposed) for the fin

To enhance heat transfer $A_t\eta_O >> A$

That is, to increase the effective area $\eta_O A_t$.

Thermal Resistance Concept:



$A=A_b+NA_{b,f}$

$R_b= t/(k_bA)$

$R_1=L_1/(k_1A)$

$R_{t,O} = 1/(hA_t\eta_O)$

$$q = \frac{T_1 - T_\infty}{\sum R} = \frac{T_1 - T_\infty}{R_1 + R_b + R_{t,O}}$$